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Abstract: A simple direct method for the recovery of surface shape from image shading is presented. The method is applicable to shading patterns obtained when a distant overhead light-source illuminates a convex or concave Lambertian surface possessing a single extremal point at which surface height has either a local maximum or local minimum. A key quality of the method is its ability to operate without explicit prerequisite information concerning the shape of the sought-after surface. Some consideration is also given to the manipulation of images of more complex surface shapes.

Key-words: Shape from shading, surfaces, Lambertian, direct method, singular point.

(Résumé : tsvp)

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Reconstruction directe de surfaces à partir des ombrages

Résumé :

Nous présentons ici une méthode simple et directe, qui permet de reconstruire une surface à partir des ombrages présents dans l'image de cette surface. Nous considérons les ombrages dans l'image d'une surface Lambertienne, concave ou convexe, obtenus pour une source lumineuse distante, située à la verticale de la surface. Cette méthode s'applique aux ombrages possédant un unique extremum pour lequel la surface admet un maximum ou un minimum local. Un des avantages majeurs de la méthode considérée est qu'elle ne nécessite pas d'information explicite quant à la forme de la surface à reconstruire. Le cas des images représentant des formes plus complexes est aussi abordé.

Mots-clé : Shape from shading, ombrages, Lambertien, méthode directe, point singulier.

1 Introduction

1.1 Motivation

The shape from shading problem may under certain assumptions be expressed mathematically as a first-order partial differential equation [7]. Much information is needed to be able to formulate such an *image irradiance* equation, the main requirement being an appropriate reflectance map [9]. Without this information, the problem cannot be posed, let alone solved. Even if the image irradiance equation can be formulated, it may be extremely difficult to solve, may have many solution shapes, or may even have no solutions [1]. A major challenge in computer vision is to develop a versatile shape from shading algorithm able to operate effectively in the absence of substantial prerequisite information [8]. Such an algorithm has so far proven elusive.

Foundational mathematical results on the initial information needed to solve the shape from shading problem appear to be at odds with the apparent human capacity to operate with less than minimal prerequisite information. While humans are not always successfully able to interpret shading patterns, they are frequently able to perceive surface shape reasonably accurately from image shading. This is apparently achieved without prior knowledge of lighting conditions, surface lustre, etc. Reconciliation is perhaps to be found in the speculation that humans make certain assumptions that prove reasonably (though not always) robust in determining shape from shading. More psychophysical work needs to be undertaken in this area to augment important contributions such as [10].

Various strategies have been employed in an attempt to reduce the prerequisite requirements of shading algorithms. The direction of the "sun" might be estimated using a statistical analysis of the image (eg. see [13]), although the available techniques appear to be far from robust [6]. Additionally, restrictions concerning the lustre of the depicted surface might be imposed resulting, for example, in the assumption that the given surface is Lambertian, or Lambertian with an added shiny component. The Lambertian model is the most commonly adopted and simplest of lustre models, and will further be employed in this work.

Perhaps most galling of all is that nearly all shape-from-shading methods need substantial prior knowledge of surface shape—the very information being sought. Thus, methods based on conventional regularisation techniques typically need to be given some surface normals of the sought-after shape. (Although the known presence of an *occluding boundary* enables such normals to be computed around the periphery of the surface image.) Local methods, such as [14], are unusual in not needing prerequisite shape information. However, a convincing case has yet to be made for the capabilities of such methods. To gain further insight into prerequisite requirements and the shape-from-shading problem, it is instructive to examine the approaches of Horn [7] and Bruckstein [2].

The foundational method of Horn requires that surface normals be given along some *initial* curve in the image. Other normals may subsequently be computed along paths sprouting from the initial curve. These paths, called *base characteristic* curves, are projections on the image domain of curves of steepest ascent or descent on the solution surface, the latter curves being themselves projections of so-called *characteristic* curves that are defined in the five-dimensional space in terms of the image, reflectance map, and initial conditions.

The method of Bruckstein requires specification of a loop in the image domain around which surface height is constant. The loop may, for example, surround the point of maximal brightness. Successive contours of constant height can then be computed by expanding outward (or contracting inward) from the initial loop.

The rather onerous initial requirements of these fundamental algorithms are formally necessary if they are to have fairly general application. Indeed, the particular form of the initial conditions will determine the extent to which the algorithms are able to sweep across the domain determining height. There are, however, certain situations in which the shape from shading problem is well posed in the absence of the initial information on shape required by these algorithms.

Foundational results on uniqueness reveal that under certain conditions shape may be determined from shading without the prior knowledge of explicit shape information, such as specific height values or surface normals (see [3], [11]). (Note, however, that implicit assumptions are always made as to the form of admissible surfaces.) By way of example, suppose that a given image is known to have been formed by illuminating a Lambertian surface with a distant overhead point source. If the image happened to be that of a wholly visible hemisphere, exhibiting a single point of maximal image brightness, the method of Horn would require an initial loop of known normals, surrounding the image point of maximal brightness, if it were to recover the shape of the hemisphere. Similarly, the method of Bruckstein would need to be given a loop in the image domain, again surrounding the image point of maximal brightness, around which surface height was known to be constant. Importantly, however, neither of these shape clues is in this situation necessary: the shape is computable without explicit initial shape data.

Our motivation in this ongoing work is to seek shape from shading algorithms, perhaps operating in a restricted environment, that require no explicit initial shape information. We propose a method of shape recovery that can be viewed as a direct variational method, in contrast with the many existing, and fundamentally different, non-direct variational methods. It should be noted that aspects of [4], [5], and [12] relate closely to our work, although the approach taken here is quite different.

1.2 Formulation

As already mentioned, the shape-from-shading problem may be expressed mathematically in terms of a first-order partial differential equation. Specifically, a function $u(x,y)$ is sought, representing surface depth in the direction of the z-axis, that satisfies the *image irradiance* equation $R(u_x, u_y) = E(x,y)$ over Ω. Here E is an image formed by (orthographic) projection of light onto a plane parallel to the xy-plane, situated above the surface, R is the reflectance map relating image brightness to surface orientation, and Ω is the image domain. In this formulation, it is implicitly assumed that light sources are infinitely far away, and internal surface reflections are disallowed.

Assume that a distant point-source illuminates a *Lambertian* surface of unit albedo. If a small surface portion with normal direction $(-u_x, -u_y, 1)$ is illuminated by a point-source of unit power in the direction determined by a unit vector (l_1, l_2, l_3), then the emitted radiance, as prescribed by Lambert's law, is given by the cosine of the angle between the two directions, namely $(l_3 - l_1 u_x - l_2 u_y)(u_x^2 + u_y^2 + 1)^{-1/2}$. Accordingly, given an image $E(x,y)$, the image irradiance equation takes the form

$$(l_3 - l_1 u_x - l_2 u_y)(u_x^2 + u_y^2 + 1)^{-1/2} = E(x,y). \tag{1}$$

If the light source is directly overhead, with $(l_1, l_2, l_3) = (0,0,1)$, then the image irradiance equation reduces to

$$(u_x^2 + u_y^2 + 1)^{-1/2} = E(x,y). \tag{2}$$

Noting that $0 < E(x,y) \leq 1$, we may safely let

$$\mathcal{E}(x,y) = E^{-2}(x,y) - 1 \tag{3}$$

and express (2) as the eikonal equation

$$u_x^2 + u_y^2 = \mathcal{E}(x,y). \tag{4}$$

Hereafter we confine ourselves to consideration of the case of an overhead light source, governed by (4). Any C^2 function satisfying (4) over Ω will be referred to as a *solution.* For a given function u, if $X \in \Omega$ is such that $u_x(X) = u_y(X) = 0$, then X will be referred to as a *singular* point of u. It is readily seen from equations (3) and (4) that any point at which E equals 1 (that is, at which the image brightness is maximal) is singular for every solution. In the sequel, given a function u, we will be particularly interested in non-degenerate singular points $X \in \Omega$ for which $\Delta(X) = u_{xx}(X)u_{yy}(X) - u_{xy}^2(X) \neq 0$. From a certain point of view, functions all of whose singular points are non-degenerate are typical. If $\Delta(X) > 0$, then always $u_{xx}(X) + u_{yy}(X) \neq 0$. A singular point X for which $\Delta(X) > 0$ and $u_{xx}(X) + u_{yy}(X) > 0$ will be called *convex*, and a singular point X such that $\Delta(X) > 0$ and $u_{xx}(X) + u_{yy}(X) < 0$ will be termed *concave.* A singular point X with $\Delta(X) < 0$ will be called *saddle.* A singular point that is either convex or convex will be called *elliptic.* When X is a singular point of u, then depending on whether $u(X)$ is convex, concave, or saddle, u will at X exhibit a local maximum, local minimum, or saddle respectively.

Note that if u is a solution to (4), then so too are functions of the form $\pm u + k$, where k is a constant. Of course, different values of the constant k in the above expression leave the shape unchanged, so we may regard $-u$ as being the only one of these surfaces different in shape to u, and this we shall call the *dual solution.*

2 An integration scheme for height recovery

In this section, we present a method that recovers surface height by sweeping outwards from an elliptical point in the image. Note that while Horn's method is unable to integrate outward from a singular point, the approach adopted here is nevertheless intimately related to the method of characteristic strips.

The essence of the method is as follows. Suppose that two points in the domain of the image are linked via a base characteristic curve that is wholly contained in the domain. An expression is derived whose integral along the base characteristic curve connecting these points equals the absolute value of the difference of the surface heights at the points in question. Moreover, the expression is such that its integral along any other curve in the domain connecting the two points is not smaller than this value. Thus the relative height of the two points can be determined by finding the minimum of integrals taken over all paths in the domain connecting them.

Given that the pattern of base characteristics is *a priori* unknown, the challenge is then to devise an implementation, utilising the method outlined above, that efficiently searches through all possible paths in the discrete domain. Below we present mathematical fundamentals and an associated computational scheme suggesting an appropriate implementation.

2.1 Mathematical derivation

The method is based on the following crucial result:

Theorem 1 *If a function u satisfies (4) in a domain Ω, and X and Y are points in Ω that can be joined by a base characteristic curve wholly contained in Ω, then the relative depth $|u(X) - u(Y)|$ between points on the graph of u, having X and Y as images, is given by*

$$|u(X) - u(Y)| = \min_{\gamma} \int_{\gamma} \sqrt{\mathcal{E}}\, dl, \tag{5}$$

where the minimum is taken over all piecewise smooth curves γ in Ω joining X and Y, and the integration is meant with respect to the standard line measure.

Proof. Remembering that $du = u_x dx + u_y dy$, we see that

$$\begin{aligned} du^2 + (u_y dx - u_x dy)^2 &= (u_x dx + u_y dy)^2 + (u_y dx - u_x dy)^2 \\ &= (u_x^2 + u_y^2)(dx^2 + dy^2). \end{aligned} \tag{6}$$

This jointly with (4) and the fact that the line element dl is defined as

$$dl^2 = dx^2 + dy^2$$

shows that

$$|du| \le \sqrt{u_x^2 + u_y^2}\sqrt{dx^2 + dy^2} = \sqrt{\mathcal{E}} dl.$$

If γ is a piecewise smooth curve joining X and Y, then, by the last inequality,

$$|u(X) - u(Y)| = \left|\int_{\gamma} du\right| \le \int_{\gamma} |du| \le \int_{\gamma} \sqrt{\mathcal{E}}\, dl.$$

Hence

$$|u(X) - u(Y)| \le \inf_{\gamma} \int_{\gamma} \sqrt{\mathcal{E}}\, dl, \tag{7}$$

where the infimum is taken over all piecewise smooth curves γ in Ω joining A and B. If γ_0 is a base characteristic joining A and B, then the form $u_y dx - u_x dy$ vanishes on γ_0, and so, by (6), $|du| = \sqrt{\mathcal{E}} dl$ on γ_0, which together with the fact that u is monotone on γ_0 implies that

$$|u(X) - u(Y)| = \int_{\gamma_0} \sqrt{\mathcal{E}}\, dl.$$

Comparison of this equation and (7) shows that the infinimum is attained on γ_0 so that the infinimum is in fact a minimum, and that this minimum satisfies (5). □

A surface having an isolated maximum (minimum) and such that every point in the surface image can be joined by a base characteristic curve with the singular point at which the surface height is maximal (minimal) will be called *convex* (*concave*). The above definition is not intrinsically geometric—it essentially depends on the direction from which the surface is viewed. Note that according to this definition, the bell-like surface shown in Figure 1a is convex.

It can easily be inferred from Theorem 1 that if a surface u generates an image with a single singular point S, then, for every point X in the image domain,

$$u(X) = u(S) - \min_{\gamma} \int_{\gamma} \sqrt{\mathcal{E}}\, dl, \tag{8}$$

whenever the surface is convex, and

$$u(X) = u(S) + \min_{\gamma} \int_{\gamma} \sqrt{\mathcal{E}}\, dl, \tag{9}$$

whenever the surface is concave.

2.2 Computational scheme

Employing the above theorem and consequences thereof, we now develop a scheme involving a local cooperative computation in which an estimate of height at some point in the domain is obtained by successive consideration of image values and height estimates in a small neighbourhood of the point. It is assumed that a base characteristic curve connects the singular point to the point under consideration.

In the algorithm, the function COMPUTE-DEPTH($\mathcal{E}$) takes as argument the transformed image $\mathcal{E}$, and returns the computed shape. The data structures U and V store depth information, each being initialised everywhere to $-\infty$, except for the singular point location, whose depth value has arbitrarily been taken to be 0. New depth values are computed from U and $\mathcal{E}$ and are written to V. We use again Ω to denote the discrete image domain, x and n represent points in the domain, and s signifies the singular point. We denote by ϵ the distance between a pixel and its nearest neighbour. The symbol $\mathcal{N}(x)$ stands for the neighbourhood of the point x, which in our implementation of the scheme is taken to be the set of the eight nearest neighbours of x in a rectangular grid. The function INTEGRATE($x, n, U, \mathcal{E}$) computes a new height value at x by subtracting from the temporary height at n the integral from n to x. As a result of a combined use of the functions COMPUTE-DEPTH($\mathcal{E}$) and INTEGRATE($x, n, U, \mathcal{E}$), a convex surface satisfying (8) is generated.

COMPUTE-DEPTH($\mathcal{E}$)
forall $x \in \Omega$ **loop**
    $U(x) \leftarrow -\infty$
end loop

```
U(s) ← 0
V ← U
loop
      forall x ∈ Ω − {s} loop
            forall n ∈ 𝒩(x) loop
                  val ← INTEGRATE(x, n, U, ℰ)
                  if val > V(x) then V(x) ← val end if
            end loop
      end loop
      if U = V then exit end if
      U ← V
end loop
return U
```

```
INTEGRATE(x, n, U, ℰ)
if n is a diagonal neighbour of x
  then Δ ← √2
  else  Δ ← 1
end if
return U(n) − Δε (√ℰ(x) + √ℰ(n)) / 2
```

At the local level, computation proceeds as follows. Consider a point at the centre of a 3×3 window of pixels in a discrete rectangular image. For each of the eight nearest neighbours that have been assigned height values, a new estimate is obtained for the height of the central point by using (8). A new value is obtained for the central point by choosing the maximum of the set comprising the above estimates, and the central point's old height value. Early on in the computation, values are computed by sweeping out from the singular point which has been assigned some arbitrary initial value. Heights across the image are recomputed as many times as is necessary to reach a steady state. In this way, complex and twisting paths are taken into account if the base characteristics exhibit such a form. When dealing with various $N \times N$ images, our implementation always reached a steady state within $2N$ iterations. Note that the scheme is amenable to parallelism as it takes a Jacobean form whereby recent updates are not immediately utilised within a given iteration, but instead are written to a new data structure.

3 Application of the integration scheme

The question now arises as to how the above integration scheme might be applied to an image for which there is known to exist a smooth solution satisfying equation (4). This simple question raises many issues.

It is useful at this stage to consider those smooth surfaces having base characteristic curves pointing outward at the periphery of the domain. We may regard such surfaces as having a *convex skirt* around their boundary [11]. Such a skirt may or may not extend far enough to become an occluding boundary. Denote by N_+, N_-, and $N_\pm$ the number of a given surface's singular points that are concave, convex, and saddle, respectively, and by N_S the total number of singular points. If the surface has a convex skirt and is defined over a simply connected domain, then

$$N_+ + N_- - N_\pm = 1$$

(see [2]). This together with the obvious equality $N_S = N_+ + N_- + N_\pm$ implies that

$$N_S = 2N_\pm + 1.$$

We thus see that smooth surfaces having a convex skirt must always have an odd number of singular points. The same result applies to the duals possessing a concave skirt.

We now characterise the combinations of singular points exhibited by smooth shapes having a convex or concave skirt. Starting with surfaces exhibiting only one singular point that is either convex or concave, with which we associate sets $\{+\}$ and $\{-\}$, respectively, we may generate new surfaces by successively adding—at the symbolic level—arbitrarily many of either $\{+\pm\}$ or $\{-\pm\}$. Thus the set $\{+ + - \pm \pm\}$ is legitimate, signifying five singular points—two convex, two saddle, and one concave. Any shape of this form will have a dual shape of type $\{+ - - \pm \pm\}$, although, in general, several shapes of somewhat different character will share the same symbolic description. The ordering of the signs being irrelevant, we adopt here a convention according to which $+$ signs go first, followed by $-$ signs, and $\pm$ signs. With this convention, successive sets are as follows:

$$\{+\},\{-\},\{++\pm\},\{+-\pm\},\{--\pm\},\{+++\pm\pm\},\{++-\pm\pm\},\{+--\pm\pm\},\{---\pm\pm\},\ldots$$

Confining our attention solely to surfaces having a convex skirt, we may eliminate some of the above, condensing the list to:

$$\{+\},\{++\pm\},\{+-\pm\},\{+++\pm\pm\},\{++-\pm\pm\},\{++++\pm\pm\pm\},\{+++-\pm\pm\pm\},\ldots$$

The duals of solutions of these kinds will have forms spanning all of those eliminated (such as $\{--\pm\}$), and will have a concave skirt.

If an image possesses no singular point, then the integration method clearly cannot be used. By way of exploration, we now consider the applicability of the previously specified method to images possessing either one or three singular points.

3.1 Images possessing one singular point

In the event that the image exhibits a single point of maximal brightness, we may utilise the integration scheme to obtain a surface. There is, however, no guarantee that the resulting surface constitutes a solution. For example, application of the integration technique will lead to an incorrect result when the given image is generated by a surface with a saddle singular point—the solution surface will then be either convex or concave, and as such will be fundamentally different from the initial surface.

It can be inferred from a result of [1] that any image can locally be modified in such a way that the resulting shading pattern is *impossible* in that it is not an image of any shape; moreover, if the initial image contains a singular point, then the modification can be made in such a fashion that the singular point will remain intact. When the integration method is applied to an impossible image with a singular point, then the resulting surface will not be smooth and/or will not satisfy the image irradiance equation.

With this remarks in mind, it is clear that the developed integration technique provides a means of finding a convex solution if we *a priori* know that such a solution exists. The solution is determined in a neighbourhood of an elliptical point covered by the emanating base characteristics. The dual concave solution is then readily obtained.

Figure 1 displays the integration method at work. When applied to a synthetic image, obtained via (4), of the bell-like surface depicted in Figure 1a, the procedure started from the isolated singular point yields, as estimate, the surface shown in Figure 1b. As we can see, the estimate approximates well the original shape.

3.2 Images possessing three singular points

In the event that the image exhibits three points of maximal brightness, any solution having a convex skirt is either of type $\{+ + \pm\}$ or of type $\{+ - \pm\}$. The question then arises as to how such a solution may be derived given that the type of each singular point is unknown.

Suppose we integrate across the whole domain from one of the singular points, thereby generating a surface. When the starting singular point is elliptic, such a surface will in general constitute a solution only over a portion of the domain. For this reason, the surface will be termed a *partial solution.* The singular point from which it was generated will be dubbed the *source.* Later, this terminology will also be applied to non-elliptic (saddle) singular points and corresponding surfaces.

In the event that the source point is elliptic, the partial solution will be an actual solution in a region covered by base characteristics emanating from the source. Surface height computed outside this region will in general fail to constitute a solution. In the event that the source point is saddle, the partial solution will fail to match a solution nearly everywhere. Correct height values will be computed only along the two base characteristics that link the saddle point with the other (elliptic) singular points.

A typical situation that arises is depicted in Figure 2. Figure 2a displays a shape with a convex skirt, whose image has three singular points. By taking successively each of these points as source, partial solutions can be built, as shown—in convex form—in Figures 2b, 2c, and 2d. None of these partial solutions constitutes a solution over the whole domain. We see, however, that surface height is correctly determined in a vast neighbourhood of each convex source. This suggests that an actual solution should be represented as a combination of the partial solutions with a convex source. That such a representation can indeed be derived will be shown later on. As a first step of a suitable construction, we identify the elliptical points, or equivalently, determine which of the three singular points is saddle. Before formulating and proving a theorem that will address this issue, we introduce some notation.

Let Ω be a simply connected domain, and $\mathcal{E}$ be a non-negative function on Ω. For every pair of points $X, Y \in \Omega$, put

$$\mathrm{d}(X,Y) = \min_{\gamma} \int_{\gamma} \sqrt{\mathcal{E}}\, dl. \tag{10}$$

The function d thus defined is a metric in Ω:

(i) $\mathrm{d}(X,Y) = 0$ if and only if $X = Y$ for all $X, Y \in \Omega$;

(ii) $\mathrm{d}(X,Y) = \mathrm{d}(Y,X)$ for all $X, Y \in \Omega$ (symmetry);

(iii) $\mathrm{d}(X,Y) \leq \mathrm{d}(X,Z) + \mathrm{d}(Z,Y)$ for all $X, Y, Z \in \Omega$ (triangle inequality).

If $S \in \Omega$ is a source in our terminology, then the corresponding partial solution U_S takes the form

$$U_S(X) = \mathrm{d}(S,X) \qquad (X \in \Omega). \tag{11}$$

For any three points X_i $(i = 1,2,3)$ in Ω and any $i = 1,2,3$, put

$$D(X_i) = \mathrm{d}(X_i, X_j) + \mathrm{d}(X_i, X_k),$$

where $j, k \in \{1,2,3\} \setminus \{i\}$.

Theorem 2 *If a function u satisfies* (4) *in Ω, has a convex skirt, and has three different critical points X_i $(i = 1,2,3)$, then the point X_i for which $D(X_i)$ is minimal is saddle.*

Proof. Suppose first that u is of type $\{+ + \pm\}$, an example of which was given in Figure 2a. Without loss of generality, we may assume that X_1 and X_2 are convex, and X_3 is saddle. Notice that the points X_1 and X_2 are not linked by a base characteristic, but are joined by a unique curve from the family of all curves that can be represented as a sum of base characteristics, namely the curve that consists of the base characteristics connecting X_3 with X_1 and X_2, respectively. The minimum in (10) corresponding to X_2 and X_3 is attained at that curve, and equals $\mathrm{d}(X_1, X_3) + \mathrm{d}(X_3, X_2)$. Let $a = \mathrm{d}(X_1, X_2)$ and $b = \mathrm{d}(X_1, X_3)$. Now, in view of the symmetry of d, $\mathrm{d}(X_1, X_2) = a + b$, and further $D(X_1) = a + b$, $D(X_2) = 2a + b$, and $D(X_3) = a + 2b$. As both a are b are positive, it is clear $D(X_3)$ is the smallest of all the $D(X_i)$ $(i = 1,2,3)$.

Suppose now that u is of type $\{+ - \pm\}$, an example of which is given in Figure 3a. Without loss of generality, we may assume that X_1 is convex, X_2 is concave, and X_3 is saddle. In the present situation all the X_i are pairwise joined by base characteristics, and so, by Theorem 1, $\mathrm{d}(X_i, X_j) = |u(X_i) - u(X_j)|$ for $i, j \in \{1,2,3\}$. But it is geometrically clear that $|u(X_1) - u(X_2)| = |u(X_1) - u(X_3)| + |u(X_3) - u(X_2)|$. In particular, we have that $\mathrm{d}(X_2, X_3) = a - b$ and $a > b$. Hence $D(X_1) = a + b$, $D(X_2) = 2a - b$, and $D(X_3) = a$, showing once again that $D(X_3)$ is the smallest of all the $D(X_i)$ $(i = 1,2,3)$. □

With Theorem 2 at hand, it is now easy to determine which of the singular points is saddle. In the next step, we find out whether or not one of the remaining points is concave, and if so, derive a solution shape. Crucial here is the observation that, for a $\{+ - \pm\}$ surface, all points in the image domain can be joined by base characteristic curves emanating from the convex singular point. Thus, if the initial surface is of type $\{+ - \pm\}$, then the partial solution with a convex source is a genuine solution, and exhibits a concavity around the concave singular point. Inspection reveals that such a concavity can be revealed neither by the other partial solution with an elliptic source in the $\{+ - \pm\}$ case, nor by the two partial solutions with an elliptic source in the $\{+ + \pm\}$ case. We thus see that the initial shape is of type $\{+ - \pm\}$ if and only if one of the partial solutions with an elliptic source exhibits a concavity, in which case that partial solution is a genuine solution.

In the event that there is no singular point is concave, a solution, necessarily of type $\{+ + \pm\}$, may be derived from two partial solutions with a convex source as follows. Offset first the height of each partial solution so that the saddle point has height zero, and next, for each image point, select the height value from the partial surface having maximal height. The resulting surface is a desired solution. The validity of this procedure is guaranteed by the following theorem:

Theorem 3 *If a function u satisfies* (4) *in Ω, has a convex skirt, and is of type $\{+ + \pm\}$ with two convex singular points X_1 and X_2, and with a saddle point X_3, then*

$$u(X) = \max\{U_{X_1}(X_3) - U_{X_1}(X), U_{X_2}(X_3) - U_{X_2}(X)\} + u(X_3). \tag{12}$$

Proof. For each $i = 1, 2$, let Ω_i be the set consisting of those points in Ω that can joined with X_i by a base characteristic. Given $i = 1, 2$, the function u is convex on Ω_i, and so, for each $A \in \Omega_i$, we have $u(A) \leq u(X_i)$ and further, by Theorem 1,

$$\mathrm{d}(X_i, A) = u(X_i) - u(A). \tag{13}$$

Let Γ be the intersection of the boundaries of Ω_1 and Ω_2. Clearly, for each $B \in \Gamma$, we have $u(B) \leq u(X_i)$ and

$$\mathrm{d}(X_i, B) = u(X_i) - u(B) \tag{14}$$

simultaneously for $i = 1$ and $i = 2$.

Let X be an arbitrary point in $X \in \Omega_1$. We first show that

$$U_{X_1}(X_3) - U_{X_1}(X) = u(X) - u(X_3). \tag{15}$$

To this end, note that X_3 belongs to Γ. Thus, in view of (14),

$$\mathrm{d}(X_1, X_3) = u(X_1) - u(X_3).$$

By (13),

$$\mathrm{d}(X_1, X) = u(X_1) - u(X). \tag{16}$$

Taking into account (11), we obtain

$$\begin{aligned} U_{X_1}(X_3) - U_{X_1}(X) &= \mathrm{d}(X_1, X_3) - \mathrm{d}(X_1, X) \\ &= u(X_1) - u(X_3) - (u(X_1) - u(X)) \\ &= u(X) - u(X_3), \end{aligned}$$

and so (15) is established.

We now prove that

$$U_{X_2}(X_3) - U_{X_2}(X) \leq u(X) - u(X_3). \tag{17}$$

Let γ be any piecewise smooth curve joining X and X_2. As X belongs to Ω_1 and X_2 lies in Ω_2, γ intersects Γ at some point Y. Let γ' be that part of γ that links X with Y, and let γ'' be the remaining part of γ linking Y with X_2. The symmetry of d and the triangle inequality guarantees that

$$\mathrm{d}(X,Y) \geq |\mathrm{d}(X_1,X) - \mathrm{d}(X_1,Y)|.$$

On account of (14),

$$\mathrm{d}(X_1,Y) = u(X_1) - u(Y),$$

which together with the previous equality and (16) implies that

$$\mathrm{d}(X,Y) \geq |u(X_1) - u(X) - (u(X_1) - u(Y))| = |u(Y) - u(X)|. \tag{18}$$

By (13) and the symmetry of d,

$$\mathrm{d}(Y,X_2) = u(X_2) - u(Y). \tag{19}$$

On the other hand, by (10),

$$\int_\gamma \sqrt{\mathcal{E}}\, dl = \int_{\gamma'} \sqrt{\mathcal{E}}\, dl + \int_{\gamma''} \sqrt{\mathcal{E}}\, dl \geq \mathrm{d}(X,Y) + \mathrm{d}(Y,X_2).$$

The latter inequality combined with (18), and (19) shows that

$$\begin{aligned}\int_\gamma \sqrt{\mathcal{E}}\, dl &\geq |u(Y) - u(X)| + u(X_2) - u(Y)\\ &\geq u(Y) - u(X) + u(X_2) - u(Y)\\ &= u(X_2) - u(X).\end{aligned}$$

Hence, in view of (10) and the arbitrariness of γ,

$$\mathrm{d}(X,X_2) \geq u(X_2) - u(X). \tag{20}$$

Now, invoking once again the fact that X_3 belongs to Γ, we infer from (14) that

$$\mathrm{d}(X_2,X_3) = u(X_2) - u(X_3).$$

Combining the latter equality with (11) and (20), we obtain

$$\begin{aligned}U_{X_2}(X_3) - U_{X_2}(X) &= \mathrm{d}(X_2,X_3) - \mathrm{d}(X_2,X)\\ &\leq u(X_2) - u(X_3) - (u(X_2) - u(X))\\ &= u(X) - u(X_3).\end{aligned}$$

Thus (17) is proved.

Now, comparing (15) and (17), we obtain (12) for all points in Ω_1. Analogously, we establish (12) for all points in Ω_2. Finally, the continuity of both sides (12) ensures the validity of the equality for all points in Γ. □

The procedure for interpreting images having three singular points of known location, but of unknown type, may now be summarised as follows:

- Locate the positions in the image of the three singular points.
- Derive three partial solutions, extending across the domain of the image, by successively treating each of the three singular points as source.
- Compute, for each partial solution, the neighbourhood sum, this being defined as the sum of the absolute height differences between the source and the other singular points.
- Locate the saddle point as the source whose partial solution generates the smallest neighbourhood sum.
- Discard the partial solution with saddle source.
- For each of the remaining (two) partial solutions, inspect the generated shape in a neighbourhood of the singular point which is neither source nor saddle. If the shape is essentially concave, then the singular point is indeed concave. Furthermore, the partial solution with the corresponding convex source is a genuine solution of type $\{+ - \pm\}$.
- In the event that there is no concave singular point, offset in height each of the partial solutions with convex source so that the saddle point has height zero, thereby registering two new partial solutions. Derive a function that is the maximum of the two newly obtained partial solutions. This will be a solution of type $\{+ + \pm\}$.
- Check whether the final estimated surface satisfies the image irradiance equation. If so, note that its dual is also a solution.

Figure 3a shows an example of a $\{+ + \pm\}$ surface, while Figure 3b shows the result obtained by employing the above procedure to its image. Similarly, the central portion of Figure 3c depicts a surface of type $\{+ - \pm\}$, with a corresponding estimate given in Figure 3d. Note that in this latter example, differences at the periphery are due to singular points, requiring a separate handling, where the initial surface is inflected. Surfaces recovered reveal relatively little error, and suggest that the method holds promise.

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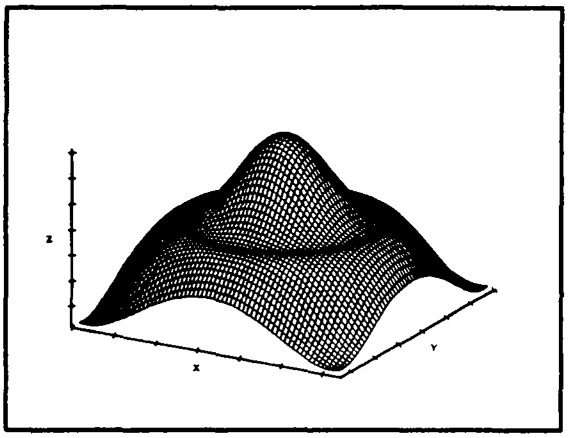
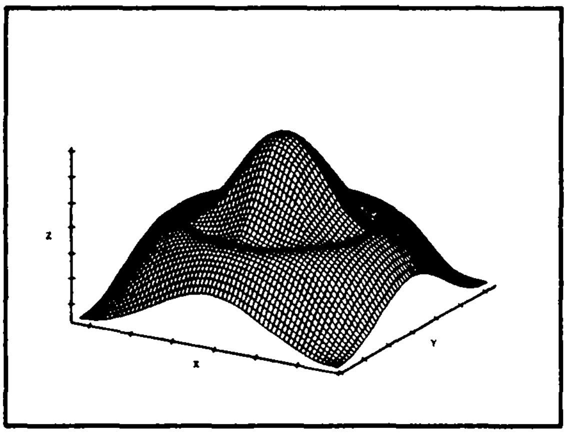
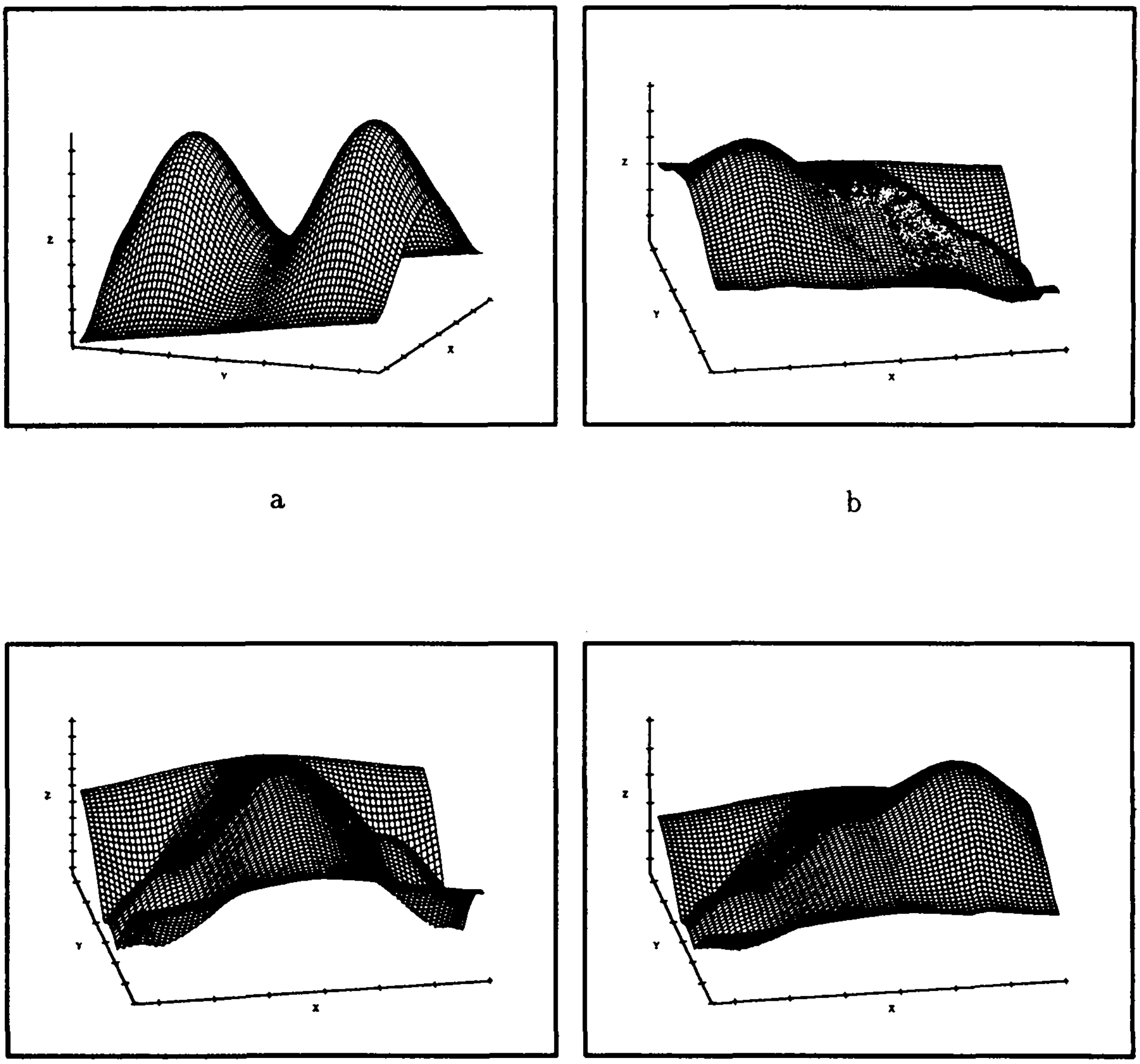


a



b

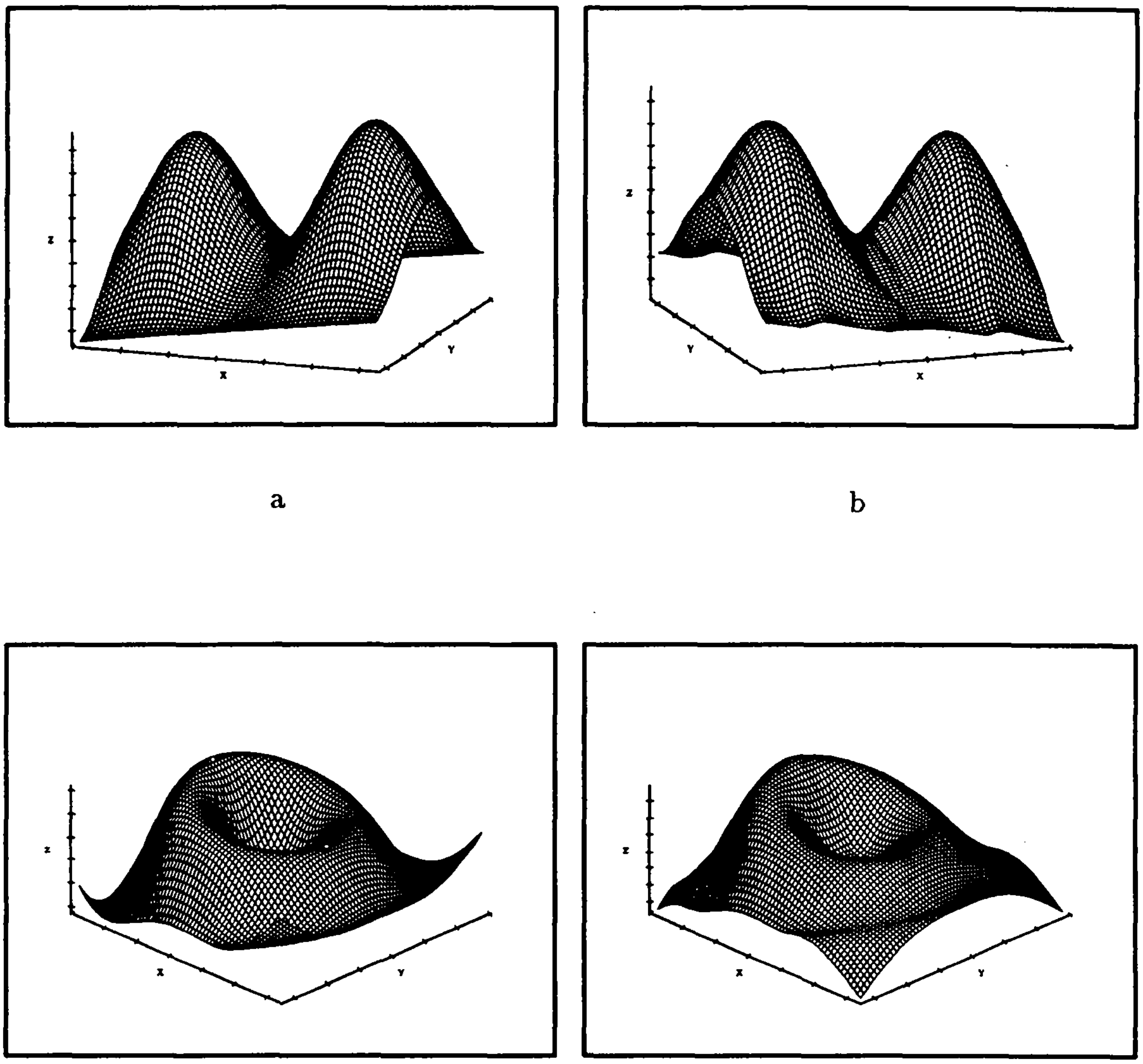
FIGURE 1: (a) depicts a bell-like surface, while (b) depicts the result of an application of the integration procedure to its image.



a b

c d

FIGURE 2: (a) depicts a surface exhibiting two peaks; given an image of (a) formed via (4), figures (b), (c) and (d) display the partial solutions obtained by treating successively each singular point as source.



a b

c d

FIGURE 3: (a) depicts a surface exhibiting two peaks, while (b) displays the estimate obtained from an image of (a); likewise (d) shows the surface derived using an image of (c).

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